PDEs

Boundary Conditions

Dirchlet: Fix **values** of the sol on $\partial\Omega$
Neumann: Fix ∇ of the sol on $\partial\Omega$
Cauchy: D + N on whole $\partial\Omega$
Mixed: D + N on disjoint parts of $\partial\Omega$

Poisson equation

$$\begin{cases} -\Delta u(\mathbf{x}) = f(\mathbf{x}) & \text{on } \Omega \\ u(\mathbf{x}) = g(\mathbf{x}) & \text{on } \partial\Omega \end{cases} \tag{1}$$

Elliptic, linear, not time dependent, infinite speed of information

Heat equation

$$\begin{cases} u_t = u_{xx} & \text{on } [0,1]\times(0,T) \\ u(0,t) = u(1,t) = 0 & \\ u(x,0) = u_0(x) & \end{cases} \tag{2}$$

Parabolic, linear, time dependent, infinite speed of information

Scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0 & \text{on } [0,1]\times(0,T) \\ \text{BC} & \\ u(x,0) = u_0(x) & \end{cases} \tag{3}$$

Hyperbolic, time dependent, finite speed of information

Scalar conservation laws
$u_t + f(u)_x = 0$

Transport equation
$u_t + a(x,t)u_x = 0$

Linear t.e.
$u_t + au_x = 0$

Burgers' equation
$u_t + uu_x = 0$ ($\Leftrightarrow$)
$u_t + \left(\frac{u^2}{2}\right)_x = 0$

Traffic model
$u_t + \left(v_{\max}(1-u)u\right)_x = 0$

Finite difference methods

1. Discretize the domain
   1D: $\Delta x > 0 \to N+2$ Points with $N = \frac{l}{\Delta x} - 1$
2. Discretize functions and derivatives
   $u_j = u(j \cdot \Delta x)$
3. Scheme from discretized PDE
   $\Rightarrow$ LSE: $AU = F$

Example I

1D Poisson on $(0,1)$
w/ homogeneous Dirichlet BC
2. $\to -\left[\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}\right]_{1\le j\le N} = f_j$
$u_0 = u_{N+1} = 0$
3. $\to U = [u_1, u_2, ..., u_N]$
$F = [f_1, f_2, ..., f_N] \cdot \Delta x^2$

$$A = \begin{bmatrix} 2 & -1 & \dots & 0 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & \dots & -1 & 2 \end{bmatrix}_{N\times N}$$

Example II

2D Poisson on $[0,1]^2$
w/ homogeneous Dirichlet BC
2. $\to$
$-\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} - \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = f_{i,j}$
$u_{0,j} = u_{N+1,j} = u_{i,0} = u_{i,M+1} = 0$
3. row-wise ordering, $\Delta x = \Delta y$
$\to U = [u_{1,1}, u_{1,2}, ..., u_{2,1}, ..., u_{N,N}]$
$F = [f_{1,1}, f_{1,2}, ..., f_{2,1}, ..., f_{N,N}] \cdot \Delta x^2$

$$A = \begin{bmatrix} B & -I & \dots & 0 \\ -I & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -I \\ 0 & \dots & -I & B \end{bmatrix}_{N^2\times N^2} \quad \text{with}$$

$I = N \times N$ Identity

$$B = \begin{bmatrix} 4 & -1 & \dots & 0 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & \dots & -1 & 4 \end{bmatrix}_{N\times N}$$

Example III

1D heat eq. on $(0,1)\times(0,T)$
/ homogeneous Dirichlet BC
2. central in space, forward in time
$\to \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0$
$u_0^n = u_{N+1}^n = 0 \quad \forall n$
$u_j^0 = u_0(x_j)$
3. Solve by explicit time stepping

Stability

$\left\|e^h\right\|_\infty = \max_{1\le j\le N}\left|(u_j - u_j^h)\right|$

Examine $-\sum_{j=1}^N u_j \mathbf{D}^+\mathbf{D}^- u_j = \sum_{j=1}^N u_j f_j$

Summation by parts & $N = \frac{1}{h}$:
$\sum_{j=1}^N (\mathbf{D}^- u_j)^2 \le \frac{1}{h}\|u\|_\infty \|f\|_\infty$

Rewrite $u_i = \sum_{j=1}^i (u_j - u_{j-1}) = h \cdot \sum_{j=1}^i 1 \cdot \mathbf{D}^- u_j$

$|u_i| \le h\left(\sum_{j=1}^i 1\right)^{\frac{1}{2}} \left(\sum_{j=1}^i (\mathbf{D}^- u_j)^2\right)^{\frac{1}{2}} \le$

$\le hN^{\frac{1}{2}}\left(\frac{\|u\|_\infty \|f\|_\infty}{h}\right)^{\frac{1}{2}} = \|u\|_\infty^{\frac{1}{2}} \|f\|_\infty^{\frac{1}{2}}$
$\Rightarrow \|u\|_\infty \le \|f\|_\infty$

Consistency

Truncation error $\tau_h^j = -\mathbf{D}^+\mathbf{D}^- u_j - f_j$
$\|\tau_h\|_\infty \le \frac{\|f''\|_\infty}{12} h^2$
$\Rightarrow \lim_{h\to 0}\|\tau_h\|_\infty = 0$

Convergence

$e_j^h = u_j - u_j^h$
$\mathbf{D}^+\mathbf{D}^- e_j^h = \mathbf{D}^+\mathbf{D}^- u_j - \mathbf{D}^+\mathbf{D}^- u_j^h$

$$\left.\begin{aligned} \mathbf{D}^+\mathbf{D}^- u_j &= -f_j \\ \mathbf{D}^+\mathbf{D}^- u_j^h &= -f_j - \mathbf{R}^j \end{aligned}\right\} -$$

$$\to \left.\begin{aligned} \mathbf{D}^+\mathbf{D}^- e_j^h &= \mathbf{R}^j \\ e_0^h = e_{N+1}^h &= 0 \end{aligned}\right\} \text{Poisson equation}$$

$\Rightarrow \left\|e^h\right\|_\infty \underset{\text{stability}}{\le} \|\mathbf{R}_h\|_\infty \underset{\text{truncation error}}{\le} Ch^2$

Lax equivalence theorem

For a <u>well-posed</u> and <u>linear</u> IVP:
A FD method is <u>consistent</u> and <u>convergent</u>
$\Leftrightarrow$ the method is <u>stable</u>

<u>*Lem. 2.4.1 (truncation error)*</u>:
$\left\|u^{\Delta x}\right\|_{\Delta x,\infty} \le \frac{1}{8}\left\|f^{\Delta x}\right\|_{\Delta x,\infty}$.

<u>*Lem. 2.4.2*</u>: If $f \in C^2(0,1)$ then:
$\left\|\tau^{\Delta x}\right\| \le \frac{\|f''\|_\infty}{12} \cdot \Delta x^2$.

Finite element method

1D Poisson equation
$$\begin{cases} -u''(x) = f(x) & \text{on } [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

- multiply by test function
- integrate over Ω
- integration by parts / Gauss-Green

$\Leftrightarrow$
Minimizing Dirichlet energy
$J(u) = \frac{1}{2}\int_0^1 |u'(x)|^2 \,\mathrm{d}x - \int_0^1 u(x)f(x)\,\mathrm{d}x$
for $u \in \mathbf{H}_0^1\left((0,1)\right)$
$f \in \mathbf{L}^2\left((0,1)\right)$
Look at critical points $J'(u) \equiv 0$
$J'(u) = \lim_{\tau \to 0} \frac{J(u+\tau v) - J(u)}{\tau} = \int_0^1 u'v' \,\mathrm{d}x - \int_0^1 fv\,\mathrm{d}x$

$\Rightarrow$ <u>Variational Formulation</u>
Find $u \in \mathbf{V}$, st.:
$\int_0^1 u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x, \forall v \in \mathbf{H}_0^1\left([0,1]\right)$

Abstract FEM

<u>Variational formulation</u>:
Find $u \in \mathbf{V}$ ($\mathbf{V}$ is Hilbert space) st:

$$\boldsymbol{a}(u,v) = \boldsymbol{l}(v) \quad \forall v \in \mathbf{V} \tag{4}$$

<u>FEM</u>:
Choose $\mathbf{V}_h \subset \mathbf{V}$ <u>finite</u> dimensional subspace $\Rightarrow$
Find $u_h \in \mathbf{V}_h$ st:

$$\boldsymbol{a}(u_h,v) = \boldsymbol{l}(v) \quad \forall v \in \mathbf{V}_h \tag{5}$$

Let $\{\varphi_j\}_{j=1}^N$ be a basis of $\mathbf{V}_h$. Then:

$$\boldsymbol{a}(u_h,\varphi_j) = \boldsymbol{l}(\varphi_j) \quad \forall 1 \le j \le N \tag{6}$$

Write $u_h = \sum_{i=1}^N u_i\varphi_i(x)$.
Then $\mathbf{AU} = \mathbf{F}$
with: $\mathbf{U} = \{u_i\}_{i=1}^N$
$\mathbf{F} = \{f_i\}_{i=1}^N \qquad f_i = \boldsymbol{l}(\varphi_i)$
$\mathbf{A} = \{A_{i,j}\}_{1 \le i,j \le N} \quad A_{i,j} = \boldsymbol{a}(\varphi_i, \varphi_j)$

If **a** is symmetric, then (16) is equivalent to:
Find $u \in \mathbf{V}$ st:

$$J(u) = \min_{v \in \mathbf{V}} \{J(v)\} \tag{7}$$

with $J(v) = \frac{1}{2}\boldsymbol{a}(v,v) - \boldsymbol{l}(v)$.
If **a** is symmetric, **A** is spd.
if **a** is not symmetric, **A** is still invertible.

Lax-Milgram theorem

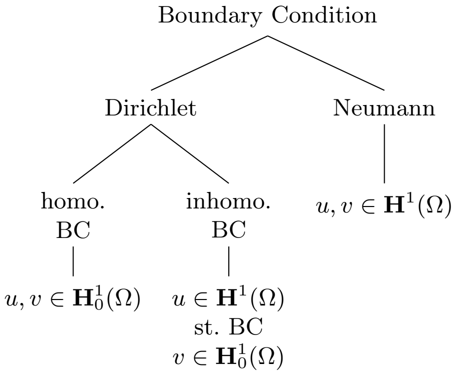
If

- $\boldsymbol{a}$ is <u>continuous</u>: $|\boldsymbol{a}(v,w)| \le \gamma\|v\|_{\mathbf{V}}\|w\|_{\mathbf{V}}$
- $\boldsymbol{a}$ is <u>coercive</u>: $\quad \boldsymbol{a}(v,v) \ge \alpha\|v\|_{\mathbf{V}}^2$
- $\boldsymbol{l}$ is <u>continuous</u>: $|\boldsymbol{l}(v)| \le \Lambda\|v\|_{\mathbf{V}}$

then the variational formulation (16) has a <u>unique</u> solution and is stable: $\|u\|_{\mathbf{V}} \le \frac{\Lambda}{\alpha}$.

Boundary conditions

Boundary conditions determine the choice of $\mathbf{V}$:



Solving Poisson equation with inhomogeneous Dirichlet BC $u(x) = g(x)$ on $\partial\Omega$ $\to u \in \mathbf{H}^1(\Omega)$:
Define $u = u_0 + \tilde{g}$
$$\tilde{g} = \begin{cases} g(x) & \text{on } \partial\Omega \\ 0 & \text{else} \end{cases}$$
$\Rightarrow u_0 \in \mathbf{H}_0^1(\Omega)$
$$\Rightarrow \begin{cases} -\Delta u_0 = f + \Delta\tilde{g} & \text{in}\Omega \\ u_0 = 0 & \text{on}\partial\Omega \end{cases}$$
In LSE: order nodes st boundary terms come last
$$\begin{bmatrix} \mathbf{A}_{00} & \mathbf{A}_{0\partial} \\ \mathbf{A}_{\partial 0} & \mathbf{A}_{\partial\partial} \end{bmatrix} \begin{bmatrix} \mathbf{U}_0 \\ \mathbf{G}_\partial \end{bmatrix} = \begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_\partial \end{bmatrix} \Rightarrow$$
$\mathbf{A}_{00}\mathbf{U}_0 = \mathbf{F}_0 - \mathbf{A}_{0\partial}\mathbf{G}_\partial$

```
setDirichletBoundary(u, interiorVertexIndices,
      ↪ vertices, triangles, bc_fct);
F −= A * u;

Sparsematrix AInterior;
igl::slice(A, interiorDofs, interiorDofs, Ainterior);

Vector FInterior;
igl::slice(F, interiorDofs, FInterior);

Eigen::SimplicialLDLT<SparseMatrix> solver;
solver.compute(AInterior);

Vector UInterior = solver.solve(FInterior);
igl::slice_into(UInterior, interiorDofs, u);
```

Neumann boundary conditions are considered when finding the variational formulation.

Error

$e_h(x) = u(x) - u_h(x)$
Exact solution:
$\boldsymbol{a}(u,v) = \boldsymbol{l}(v) \quad \forall v \in \mathbf{V}$
$\therefore \boldsymbol{a}(u,v) = \boldsymbol{l}(v) \quad \forall v \in \mathbf{V}_h$
FEM soluton:
$\boldsymbol{a}(u_h,v) = \boldsymbol{l}(v) \quad \forall v \in \mathbf{V}_h$

Subtract $\to$ Galerkin orthogonality:

$$\boldsymbol{a}(e_h,v) = 0 \quad \forall v \in \mathbf{V}_h \tag{8}$$

Choose $w = u_h - v$ for a $v \in \mathbf{V}_h$, write $\|e_h\|_{\mathbf{H}_0^1}^2 \le \frac{1}{\alpha}\boldsymbol{a}(e_h,e_h) + \boldsymbol{a}(e_h,w)$ and use Galerkin orthonormality to show Cea's inequality:
$\|e_h\|_{\mathbf{V}} \le \frac{\gamma}{\alpha}\|u - v\|_{\mathbf{V}} \quad \forall v \in \mathbf{V}_h$
$\therefore$ FEM error is bounded by interpolation error.

$$\text{1D:} \begin{cases} p\text{-order basis functions} \\ \mathbf{I}_h u(d) = u(d) \quad \forall d \quad (\text{for } p{=}2\text{: } d{=}\text{Vertex,Midpoint}) \\ \|e_h\|_{\mathbf{H}_0^1} \le Ch^{\min\{p+1,k\}-1}\|u\|_{\mathbf{H}^k} \\ \|e_h\|_{\mathbf{L}^2} \le Ch^{\min\{p+1,k\}}\|u\|_{\mathbf{H}^k} \\ \text{with } p\text{: order of FEM basis} \\ \qquad k\text{: "nicenessöf solution} \\ \text{Ex: Piecewise linear basis } (p=1) \\ \|e_h\|_{\mathbf{H}_0^1} \le Ch\|u\|_{\mathbf{H}^2} \\ \|e_h\|_{\mathbf{L}^2} \le Ch^2\|u\|_{\mathbf{H}^2} \end{cases}$$

$$\text{2D:} \begin{cases} \|e_h\|_{\mathbf{H}_0^1} \le C(c_{min}, c_{max})h\left\|\mathrm{D}^2 u\right\|_{\mathbf{L}^2} \\ \text{If } \Omega \text{ is convex:} \\ \left\|\mathrm{D}^2 u\right\|_{\mathbf{L}^2} \le C\|f\|_{\mathbf{L}^2} \end{cases}$$

Implementation

Implement $\mathbf{AU}=\mathbf{F}$

Integrate element-wise

Locality of basis: $A_{i,j} = \boldsymbol{a}(\varphi_i,\varphi_j) = \sum_m \boldsymbol{a}_{k_m}(\varphi_i\varphi_j)$

$f_i = \boldsymbol{l}(\varphi_i) = \sum_m \boldsymbol{l}_{k_m}(\varphi_i)$

Summands only nonzero if i,j on the same triangle $k_m \to$ compute local element stiffness matrix / load vector and assemble global stiffness matrix / load vector.

Integrate on reference triangle

Use affine linear map from reference triangle $\hat{k}=\{(0,0),(1,0),(0,1)\}$ to real triangle $k=\{\vec{N}_A,\vec{N}_B,\vec{N}_C\}$:

$\mathbf{J}_k = \begin{bmatrix}\vec{N}_B-\vec{N}_A & \vec{N}_C-\vec{N}_A\end{bmatrix}_{d\times d}$

$\Phi_k(\hat{\mathbf{x}}) = \vec{N}_A + \mathbf{J}_k\hat{\mathbf{x}}$

$\Phi_k^{-1}(\mathbf{x}) = \mathbf{J}_k^{-1}(\mathbf{x}-\vec{N}_A)$

φ_α^k: shape function at vertex α of triangle k

$\hat{\varphi}_\alpha$: shape function at vertex α of refrence triangle

$\varphi_\alpha^k = \hat{\varphi}_\alpha(\Phi_k^{-1}(\mathbf{x}))$ $\qquad \hat{\varphi}_\alpha(\hat{\mathbf{x}}) = \varphi_\alpha^k(\Phi(\hat{\mathbf{x}}))$

$\nabla\varphi_\alpha^k = \mathbf{J}_k^{-T}\nabla\hat{\varphi}(\mathbf{x})$ $\qquad \nabla\hat{\varphi}_\alpha(\hat{\mathbf{x}}) = \mathbf{J}^T\nabla\varphi_\nabla^k(\hat{\mathbf{x}})$

$\to A_{i,j} = \sum_m \hat{\boldsymbol{a}}_{k_m}(\hat{\varphi}_i,\hat{\varphi}_j)$

$f_i = \sum_m \hat{\boldsymbol{l}}_{k_m}(\hat{\varphi}_i)$

Example:

$\begin{cases}\Delta u = f & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\end{cases}$

$A_{i,j} = \int_\Omega \nabla\varphi_i(x)\cdot\nabla\varphi_j(x)\,\mathrm{d}x$

$= \sum_k \int_k \nabla\varphi_i(x)\cdot\nabla\varphi_j(x)\,\mathrm{d}x$

$= \sum_k \int_{\hat{k}} (\mathbf{J}_k^{-T}\nabla\hat{\varphi}_i(\hat{\mathbf{x}}))\cdot(\mathbf{J}_k^{-T}\nabla\hat{\varphi}_j(\hat{\mathbf{x}}))|\det\mathbf{J}_{k_m}|\,\mathrm{d}\hat{\mathbf{x}}$

$f_i = \int_\Omega f(x)\varphi_i(x)\,\mathrm{d}x$

$= \sum_k \int_k f(x)\varphi_i(x)\,\mathrm{d}x$

$= \sum_k \int_{\hat{k}} f(\Phi_k(\hat{\mathbf{x}}))\varphi_i(\hat{\mathbf{x}})|\det\mathbf{J}_k|\,\mathrm{d}\hat{\mathbf{x}}$

Pseudocodes

Compute element stiffness matrix $\mathbf{A}^k$

For $\alpha=0,1,2$

For $\beta=0,1,2$

$A_{\alpha,\beta}^k = \text{integrate}(\mathbf{J}_k^{-T}\nabla\hat{\varphi}_\alpha\cdot\mathbf{J}_k^{-T}\nabla\hat{\varphi}_\beta|\det\mathbf{J}_k|)$

Assemble global stiffness matrix $\mathbf{A}$

For each triangle k

$\mathbf{A}^k =$ compute local stiffness matrix

$\mathbf{I}=(i,j,k)$ indices of vertices of k

For $\alpha=0,1,2$

For $\beta=0,1,2$

$A(\mathbf{I}(\alpha),\mathbf{I}(\beta)) += A_{\alpha,\beta}^k$

Compute element load vector $\mathbf{F}^k$

For $\alpha=0,1,2$

$F_\alpha^k = \text{integrate}(\hat{\varphi}_\alpha f(\Phi_k(\hat{\mathbf{x}}))|\det\mathbf{J}_k|)$

Assemble global load vector $\mathbf{F}$

For each triangle k

$\mathbf{F}^k =$ compute element stiffness matrix

$\mathbf{I}=(i,j,k)$ indices of vertices of k

For $\alpha=0,1,2$

$F(\mathbf{I}(\alpha)) += F_\alpha^k$

Chapter 4: FEM

4.1 1D Poisson

4.1.0 The energy method

Lem. Poincaré:

$$\int_0^1 |u(x)|\,\mathrm{d}x \le \int_0^1 |u'(x)|^2\,\mathrm{d}x \tag{9}$$

Furthermore, the estimate (22) and the Poincaré inequality (9) can be written as: $||u||_{L^\infty} \le ||u||_{H_0'}$ and $||u||_{L^2} \le ||u||_{H_0}$.

We can further summarize the estimates (22)-(24) as:

$$\begin{aligned} ||u||_{L^2} &\le ||f||_{L^2}\\ ||u||_{L^\infty} \overset{\text{Poincaré}}{\le} ||u||_{H_0'} &\le ||f||_{L^2}\\ ||u||_{L^\infty} &\le ||f||_{L^2}\end{aligned} \tag{10}$$

Uniqueness: An immediate consequence of the energy estimates (10) is the following Lemma **Lem. Uniqueness**: The solutions of the Poisson equation (2) are unique.

4.1.1 Variational formulation

4.12 Finite element formulation

Ex. S4A1a: The evolution of the temperature distribution $u=u(\mathbf{x},t)$ in a homogeneous "2D body"(occupying the space $\Omega\subset\mathbb{R}^2$) with convective cooling is modeled by the linear second-order parabolic initial-boundary value problem (IBVP) with flux (spatial) boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= 0 && \text{in } \Omega\times[0,T],\\ -\nabla u\cdot n &= \gamma u && \text{on } \partial\Omega\times[0,T],\\ u(\mathbf{x},0) &= u_0(\mathbf{x}) && \text{in } \Omega,\end{aligned} \tag{11}$$

with $\gamma>0$. For the semi-discretization of (11) we employ linear finite elements on a triangular mesh $\mathcal{M}$ of Ω with polygonal boundary approximation. Derive the spatial variational formulation of the form $m(\dot{u},v)+a(u,v)=l(v)$ for (11), with suitable bilinear forms a amd m, and a linear form l. Specify the function spaces for $u(t,\cdot)$ and the test function v.

Solution. find $u\in H^1(\Omega)$, s.t $\forall\, v\in H^1(\Omega)$

$$\underbrace{\int_\Omega \dot{u}v\,\mathrm{d}x}_{=m(\dot{u},v)} + \underbrace{\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_{\partial\Omega}\gamma uv\,\mathrm{d}S}_{=a(u,v)} = \underbrace{0}_{=l(v)}.$$

With $V=H_0^1([0,1])$, the variational formulation of the one-dimensional Poisson equation is: Find $u\in V$ such that for all $v\in V$ it holds that:

$$\int_0^1 u'(x)v'(x)\,\mathrm{d}x = \int_0^1 v(x)f(x)\,\mathrm{d}x. \tag{12}$$

Using the notation $(g,h) := \int_0^1 g(x)h(x)\,\mathrm{d}x$, we may write (12) concisely as

$$(u',v') = (f,v) \quad \forall\, v\in V.$$

$$V^h = \left\{\begin{array}{l} w:[0,1]\to\mathbb{R}s.t.w \text{ is continuous},\\ w(0)=w(1)=0, \text{ and } w|_{[(j-1)h,jh]}\\ \text{is linear } \forall\, j\in\{1,\dots,N+1\}\end{array}\right\}$$

Find $u_h\in V^h$ such that for all $v\in V^h$ it holds that $(u_{h'}',v')=(f,v)$ (FEM).

4.1.3 Concrete realisation of FEM

Definition hat/tent functions:

$$\varphi_j(x) = \begin{cases}\frac{x-x_{j-1}}{h} & x\in[x_{j-1},x_j),\\ \frac{x_{j+1}-x}{h} & x\in[x_j,x_{j+1}),\\ 0 & \text{otherwise.}\end{cases}$$

Then $\varphi_j(x_i)=\delta_{ij}$ and

$$\varphi_j(x)' = \begin{cases}\frac{1}{h} & x\in[(j-1)h,jh),\\ -\frac{1}{h} & x\in[jh,(j+1)h],\\ 0 & \text{otherwise.}\end{cases}$$

Given a function $w\in V^h$, it clearly holds that

$$w(x) = \sum_{j=1}^N w_j\varphi_j(x), \quad \text{with } w_j = w(x_j).$$

$$\begin{aligned}\langle Aw,w\rangle &= \sum_{i,j=1}^N w_iA_{ij}w_j = \sum_{i,j=1}^N w_i(\varphi_i',\varphi_j')w_j\\ &= \left(\sum_{i=1}^N w_i\varphi_i', \sum_{j=1}^{\cdot N} w_j\varphi_j'\right)\\ &= (\bar{w}',\bar{w}') \overset{\text{if } \bar{w}\neq 0}{>} 0.\end{aligned}$$

4.1.4 Computing stiffness matrix and vectorload

For all $i,j=1,\dots,N$: $A_{ij}=(\varphi_i',\varphi_j')=$

$$= \begin{cases} 0 & |i-j|>1,\\ -\frac{1}{h} & |i-j|=1,\\ \frac{1}{h^2}\int_{x_{j-1}}^{x_{j+1}} 1\,\mathrm{d}x = \frac{2}{h} & |i-j|=0.\end{cases}$$

Therefore, the matrix A is given by

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \dots & 0\\ -1 & 2 & -1 & & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & & -1 & 2 & -1\\ 0 & \dots & 0 & -1 & 2\end{bmatrix}$$

4.1.5 Convergence Analysis

Definition Error Function: $e_h := u-u_h$.

Galerkin orthog.: $(u',v')=(f,v)$ and $(u_h',v')=(f,v) \implies (e_h,v')=0$.

Define; $w=u_h-v\in V^h$. Then

$$\begin{aligned}||e_h||_{H_0^1([0,1])}^2 &= \int_0^1 |e_h'(x)|^2\,\mathrm{d}x = (e_h',e_h')\\ &= (e_h',e_h') + (e_h',w') \text{ (because of Gal. orth.)}\\ &= (e_h',(e_h'+w')) = (e_h',(u-v)')\\ &= \int_0^1 e_h'(x)(u'(x)-v'(x))\,\mathrm{d}x\\ &\overset{\text{CS}}{\le} \left(\int_0^1 |e_h'|^2\,\mathrm{d}x\right)^{1/2}\cdot\left(\int_0^1 |u'-v'|^2\,\mathrm{d}x\right)^{1/2}\\ &= ||e_h||_{H_0^1([0,1])}\cdot||u-v||_{H_0^1([0,1])}.\end{aligned}$$

Hence, for all $v\in V^h$ it holds that

$$||e_h||_{H_0^1([0,1])} \le ||u-v||_{H_0^1([0,1])}.$$

Cea's Lemma: In other words, for all $v\in V^h$ it holds that

$$||u-u_h||_{H_0^1([0,1])} \le ||u-v||_{H_0^1([0,1])}. \tag{13}$$

Define such a $v =: I_hu$. Then

$$||u-I_hu||_{H_0^1([0,1])} \le Ch, \tag{14}$$

where C is a constant that depends on $\max_{0\le y\le 1}|u''(y)|$. (13) and (14) imply that

$$||e_h||_{H_0^1([0,1])} \le ||u-u_h||_{H_0^1([0,1])} \le Ch.$$

Thus, the FEM converges to the exact solution as $h\to 0$ and the rate of convergence in the $H_0^1([0,1])$

norm is at least 1. In order to obtain a corresponding error erstimate for the $L^2([0,1])$ norm, we can use the Poincaré inequality (9) to obtain

$$||u-u_h||_{L^2([0,1])} \leq Ch. \quad \text{CRUDE!}$$

Aubin-Nitze trick:
Duel problem: Find $\varphi \in H_0^1$ such that

$$\begin{aligned} -\varphi''(x) &= e_h(x) \\ \varphi(0) &= \varphi(1) = 0 \end{aligned} \tag{15}$$

$$\begin{aligned}
||e_h||_{L^2}^2 &= \int_0^1 e_h(x)e_h(x)\,\mathrm{d}x \\
&\overset{(15)}{=} -\int_0^1 \varphi''(x)e_h(x)\,\mathrm{d}x \\
&\overset{\text{i.b.p}}{=} \int_0^1 \varphi'(x)e_h'(x)\,\mathrm{d}x \\
&= (e_h', \varphi') - (e_h', \underbrace{(I_h\varphi)'}_{=0 \text{ by Gal.Orth}}) \\
&= (e_h', (\varphi' - (I_h\varphi))') \\
&\overset{C.S.}{\leq} ||e_h||_{H_0^1}||\varphi - I_h\varphi||_{H_0^1} \\
&\overset{\text{Int.err.}}{\leq} C_1 h \cdot C_2 h ||\varphi''||_{L^2} \leq Ch^2||\varphi''||_{L^2} \\
&\overset{(15)}{\leq} Ch^2||e_h||_{L^2}
\end{aligned}$$

$\implies ||e_h||_{L^2} \leq Ch^2$ which is a better error estimate.

4.2.4. Numerical Experiments in 2D

Rem. 4.2.4:
2D: $h = O(N^{-1/2})$
1D: $h = O(N^{-1})$
$\Leftrightarrow EOC_{H_0^1}(h) = 2EOC(N)$

Rem.: $u \in H_0^1 \Rightarrow u \in L^2$
Von Neumann BC:
Find $u \in H^1(\Omega)$ s.t.:
$\int_\Omega \langle \nabla u, \nabla v\rangle\,\mathrm{d}x + \int_\Omega uv\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x + \int_\Gamma gv\,\mathrm{d}S$
$\forall v \in H^1(\Omega)$
$\overset{\text{i.b.p.}}{\Leftrightarrow} \ldots \Leftrightarrow$
$\int_\Omega(-\Delta + u - f)v\,\mathrm{d}x = \int_\Gamma(-\frac{\partial u}{\partial v} + g)v\,\mathrm{d}S$

1. $H_0^1(\Omega) \subset H^1(\Omega)$, choose $v \in H_0^1(\Omega)$:
   $\Rightarrow \int_\Omega(-\Delta + u - f)v\,\mathrm{d}x = 0, \qquad \forall v \in H_0^1(\Omega)$
   $\Rightarrow -\Delta + u = f, \qquad \forall x \in \Omega.$
2. $\int_\Gamma(-\frac{\partial u}{\partial v} + g)v\,\mathrm{d}S = 0, \qquad \forall v \in H^1(\Omega)$
   $\Rightarrow \frac{\partial u}{\partial v} = g, \qquad$ on Γ (boundary)

Triangulation

Split Ω in non-overlapping triangles with no hanging nodes.
$h = \max\limits_{k \in T_h}\{\text{diam}(k)\}$; $\text{diam}(k)$ for ex. longest edge of k

Shape-regular triangulation

T_h is shape regular if $\exists$ constants $c_{min},\ c_{max}$, s.t.
$c_{min}h \leq \text{diam}(k) \leq c_{max}h \ \ \forall k \in T_h$
$\Leftrightarrow$ Triangles in T_h are of similar shape and size.

Representing triangulations

List of vertices:

$$Z = \begin{bmatrix} v_1^x & v_2^x & \ldots & v_{N_v}^x \\ v_1^y & v_2^y & \ldots & v_{N_v}^y \\ v_1^z & v_2^z & \ldots & v_{N_v}^z \end{bmatrix} \in \mathbb{R}^{\dim \times N_v}$$

List of triangles:

$$T = \begin{bmatrix} i_1 & i_2 & \ldots & i_{N_T} \\ j_1 & j_2 & \ldots & j_{N_T} \\ k_1 & k_2 & \ldots & k_{N_T} \end{bmatrix} \in \mathbb{N}_+^{\text{dof} \times N_T}$$

Coordinates of the i-th vertex of the j-th triangle: $Z(\cdot, T(i,j))$.

Heat equation

$$\begin{cases} u_t = u_{xx} & \text{on } (0,1)\times(0,T) \\ u(0,t)=u(1,t)=0 \\ u(x,0)=u_0(x)\end{cases} \tag{16}$$

Fourier's method

Separation of variables:
$\to u(x,t)=\sum_{k=-\infty}^{\infty} a_k \sin(k\pi x) e^{-(k\pi)^2 t}$
Find a_k for given u_0 by:
$a_k = 2\int_0^1 u_0(x)\sin(k\pi x)\,\mathrm{d}x$
Algorithm for Fourier's method (Spectral method):
1. Given u_0, obtain a_k by quadrature
2. Truncate the initial expansion
$u_K^0(x)=\sum_{k=-K}^{K} a_k\sin(k\pi x)$
3. Evaluate approximate solution
$u_K(x,t)=\sum_{k=-K}^{K} a_k \sin(k\pi x)e^{(-k\pi)^2 t}$

Numerical schemes

$\lambda = \frac{\Delta t}{\Delta x^2}$
$E^\Delta_{\mathbf{L}^2} = \Delta x \sum_j \left|u_j^n - u(x_j,t^n)\right|^2$
Truncation error: "plug exact solution into numerical scheme", has the same scaling as $E^\Delta_{\mathbf{L}^2}$.

Rem. spatial central and temporal forward diff:
$|u_{xx}(x_j,t^n) - \frac{u^n_{j+1}-2u^n_j+u^n_{j-1}}{\Delta x^2}| \le C\Delta x^2$
$|u_t(x_j,t^n) - \frac{u_j^{n+1}-u_j^n}{\Delta t}| \le C\Delta t$
Explicit FD
Central in space, forward in time
$\left[\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{u^n_{j+1}-2u^n_j+u^n_{j-1}}{\Delta x^2}\right]_{2\le j\le N-1}$
$\frac{u_1^{n+1}-u_1^n}{\Delta t} = \frac{u_2^n-2u_1^n}{\Delta x^2}$
$\frac{u_N^{n+1}-u_N^n}{\Delta t} = \frac{u_{N-1}^n-2u_N^n}{\Delta x^2}$
$u_j^0 = u_0(x_j)$
Update Form
$u_j^{n+1} = \lambda u^n_{j+1} + (1-2\lambda)u_j^n + \lambda u^n_{j-1}$
$u_1^{n+1} = \lambda u_2^n + (1-2\lambda)u_1^n$
$u_N^{n+1} = \lambda u_{N-1}^n + (1-2\lambda)u_N^n$
stable iff $\lambda\le\frac12$
Truncation error:
$\tau_j^n = \frac{u_j^{n+1}-u_j^n}{\Delta t} - \frac{u^n_{j-1}-2u^n_j+u^n_{j+1}}{\Delta x^2}$
$E^\Delta \le C(\Delta t+\Delta x^2)\le C(\Delta x^2)$
Implicit FD
Central in space at t^{n+1}, backward in time
$\left[\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{u^{n+1}_{j+1}-2u^{n+1}_j+u^{n+1}_{j-1}}{\Delta x^2}\right]_{2\le j\le N-1}$
$\frac{u_1^{n+1}-u_1^n}{\Delta t} = \frac{u_2^{n+1}-2u_1^{n+1}}{\Delta x^2}$
$\frac{u_N^{n+1}-u_N^n}{\Delta t} = \frac{u_{N-1}^{n+1}-2u_N^{n+1}}{\Delta x^2}$
$u_j^0=u_0(x_j)$
Update form
$u_j^n = -\lambda u^{n+1}_{j+1} + (1+2\lambda)u_j^{n+1} - \lambda u^{n+1}_{j-1}$
$u_1^n = -\lambda u_2^{n+1} + (1+2\lambda)u_1^{n+1}$
$u_N^n = -\lambda u_{N-1}^{n+1} + (1+2\lambda)u_N^{n+1}$
$\Rightarrow$ LSE: $\mathbf{AU^{n+1}=U^n}$

$$\mathbf{A}=\begin{bmatrix} 1+2\lambda & -\lambda & \dots & 0 \\ -\lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\lambda \\ 0 & \dots & -\lambda & 1+2\lambda \end{bmatrix}_{N\times N}$$

unconditionally stable
$E^\Delta\le C(\Delta t+\Delta x^2)$
Crank-Nicholson scheme
Linear combination of explicit and implicit
$\left[\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{u^n_{j+1}-2u^n_j+u^n_{j-1}+u^{n+1}_{j+1}-2u^{n+1}_j+u^{n+1}_{j-1}}{2\Delta x^2}\right]_{2\le j\le N-1}$
$\frac{u_1^{n+1}-u_1^n}{\Delta t} = \frac{1}{2\Delta x^2}(u_2^n-2u_1^n+u_2^{n+1}-2u_1^{n+1})$
$\frac{u_N^{n+1}-u_N^n}{\Delta t} = \frac{1}{2\Delta x^2}(u_{N-1}^n-2u_N^n+u_{N-1}^{n+1}-2u_N^{n+1})$
Update form:
$\left[\frac{\lambda}{2}u^{n+1}_{j+1}+(1+2\lambda)u_j^{n+1}-\frac{\lambda}{2} u^{n+1}_{j-1} = \frac{\lambda}{2} u^n_{j+1}-(1-\lambda)u^n_j+\frac{\lambda}{2} u^n_{j-1}\right]_{2\le j\le N-1}$
$-\frac{\lambda}{2} u_2^{n+1}+(1+\lambda)u_1^{n+1} = \frac{\lambda}{2} u_2^n + (1-\lambda)u_1^n$
$-\frac{\lambda}{2} u_{N-1}^{n+1}+(1+\lambda)u_N^{n+1} = \frac{\lambda}{2} u_{N-1}^n+(1-\lambda)u_N^n$
$\Rightarrow$ LSE $\mathbf{AU^{n+1}=BU^n}$

$$\mathbf{A}=\begin{bmatrix} 1+\lambda & -\frac{\lambda}{2} & \dots & 0 \\ -\frac{\lambda}{2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & \dots & -\frac{\lambda}{2} & 1+2\lambda \end{bmatrix}_{N\times N}$$

$$\mathbf{B}=\begin{bmatrix} 1-\lambda & \frac{\lambda}{2} & \dots & 0 \\ \frac{\lambda}{2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & \dots & \frac{\lambda}{2} & 1-2\lambda \end{bmatrix}_{N\times N}$$

CN is unconditionally stable, but can generate small spurious extrema when $\lambda>1$.
$E^\Delta\le C(\Delta t^2+\Delta x^2)$
FEM
Find $u_h\in\mathbf{V}_h\subset\mathbf{H}^1_0(\Omega)$ st
$\Delta t\big((u_h^{n+1})_x,v_x\big)_{\mathbf{L}^2}+\big(u_h^{n+1},v\big)_{\mathbf{L}^2} = \big(u_h^n,v\big)_{\mathbf{L}^2}\quad\forall v\in\mathbf{V}_h$

Energy principle

$E(t)=\frac12\int_\Omega u^2(x,t)\,\mathrm{d}x$
$E^n(t)=\frac{\Delta x}{2}\sum_j(u_j^n)^2$

$$E(t)\le E(0) \tag{17}$$

Show $\frac{\mathrm{d}E}{\mathrm{d}t}\le 0$ by PI+BC.
Consequences
Stability
Let u,v solve the heat equation. Then $w=u-v$ solves it as well, so by the energy principle:
$\int_0^1|u-v|^2\,\mathrm{d}x \le \int_0^1 |u_0-v_0|^2\,\mathrm{d}x$
Pertubations get smaller
Uniqueness
If $u,\tilde u$ both solve the heat equation with the same initial data, then by stability:
$u(x,t)=\tilde u(x,t)\quad\forall(x,t)$
$w:=u-\tilde{(u)}\Rightarrow w=u_0(x)-u_0(x)=0\Rightarrow u=\tilde u$
(initial condition & energy fct)
Consistency of Fourier's method
As $K\to\infty$, $u_K^0\to u_0$ and by stability, $u_K\to u$.
Stability of numerical schemes
Stable in the sense that they satisfy the energy principle:

Explicit	iff $\lambda<\frac12$
Implicit	unconditional
CN	unconditional

This can be shown by manipulating the numerical scheme in it's operator representation.

Maximum principle

$\bar u^n = \max_j\{u_j^n\}$
$\underline{u}^n = \min_j\{u_j^n\}$

$\min\{0,u_0(x)\}\le u(x,t)\le\max\{0,u_0(x)\}\quad\forall(x,t)$
$\Rightarrow \underline{u}^n\le\underline{u}^{n+1}\le\bar u^{n+1}\le\bar u^n$

Show by contradiction, $u_{xx}(x_0,t_0)\overset{!}{<}0$ and the heat equation that the strict maximum of $u(x,t)$ can't lie in the interior or at $t=T$. Remove the possibility for a non-strict maximum by arguing similarly about $u^\epsilon=u(x,t)-\epsilon t$. And in the end let $\epsilon\to0$.

Consequences Stability of numerical schemes
Stable in the sense that they satisfy the maximum principle:

Explicit	iff $\lambda<\frac12$
Implicit	unconditional
CN	iff $\lambda<1$

This can be shown by manipulating the scheme in it's update form.

Computational work

$W\sim$ mesh points x time steps, i.e. $\frac{1}{\Delta x}\cdot\frac{1}{\Delta t}=\frac{1}{\Delta x^3}$ (expl. case)
Explicit (CFL: $\Delta x^2\sim\Delta t\ll1$):
$E^\Delta_{ex}\sim\Delta t+\Delta x^2\sim\Delta x^2$
$W_{ex}\sim\frac{1}{\Delta x\Delta t}\sim\frac{1}{\Delta x^3}\quad\left(\frac{1}{\Delta x^{d+2}}\right)$
$\Rightarrow\Delta x\sim W^{-1/3}$
$\to E^\Delta_{ex}\sim W^{-\frac23}$
Implicit ($\Delta t\sim\Delta x\ll1$):
$E^\Delta_{im}\sim\Delta t+\Delta x^2\sim\Delta x$
$W_{im}\sim\frac{1}{\Delta x\Delta t}\sim\frac{1}{\Delta x^2}\quad\Rightarrow\Delta x\sim W^{-1/2}$
$\to E^\Delta_{im}\sim W^{-\frac12}$
Crank-Nicholson ($\Delta t\sim\Delta x\ll1$):
$E^\Delta_{CN}\sim\Delta t^2+\Delta x^2\sim\Delta x^2$
$W_{ex}\sim\frac{1}{\Delta x\Delta t}\sim\frac{1}{\Delta x^2}\quad\Rightarrow\Delta x\sim W^{-1/2}$
$\to E^\Delta_{ex}\sim W^{-1}$
$\to$ for the same amount of work, $E^\Delta_{im}>E^\Delta_{ex}>E^\Delta_{CN}$.

Scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u_0(x) \end{cases} \qquad (101)$$

Method of characteristics

For a scalar conservation law written as
$\begin{cases} u_t + a(u,x,t)u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$
choose the ansatz
$\begin{cases} \dot{x}(t) = a(u,x(t),t) \\ x(0) = x_0 \end{cases}$
to find curves on which $u(x,t)$ is constant.
$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t),t) = u_t + u_x\dot{x} = 0$
$\Rightarrow u(x(t_1),t_1) = u(x(t_2),t_2) \quad \forall (t_1,t_2)$
The solution is then
$u(x(t),t) = u(x(0),0) = u_0(x_0)$

Transport equation

Scalar conservation law with $f(u)_x = a(x,t)u_x$.

Solution via characteristics
Linear transport equation: $a \in \mathbb{R}$
$\dot{x}(t) = a$ (equation of characteristics) [solve this ODE]
$\rightarrow x(t) = at + x_0, x_0 = x(0)$ [only for const. a, solve for t to do the t-x plane plot!]
$\rightarrow u(x,t) = u_0(x_0) = u_0(x - at)$

Energy principle
For $\lim_{x\to\pm\infty} u(x,t) = 0 \ \forall t, \ \|a_x\|_\infty < \infty$
$\frac{dE(t)}{dt} \le \|a_x\|_\infty E(t) \Leftrightarrow E(t) \le E(0)e^{\|a_x\|_\infty t}$
Proof: multplying (101) by u, chain rule, product rule, integrate over space, decay to zero at infinity, regularity of a, Gronwall's inequality
$\rightarrow$ FD central in space, forward in time is unconditionally unstable.

Scheme
$a(x,t)$ dictates direction and speed of propagation of information.
Characteristics $\rightarrow u(x,t) = u_0(x - at)$ for $a \in \mathbb{R}$
$[a > 0 \quad \longrightarrow] \quad | \quad [a < 0 \quad \longleftarrow]$
Domain of dependence $\subseteq$ numerical stencil
$u(x, t+\Delta t) = u_0(x - a(t+\Delta t)) =$
$u_0((x - a\Delta t) - at) = u(x - a\Delta t, t)$ if $a \in \mathbb{R}$

$\rightarrow$ **Simple centered FD scheme**
$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a(u_{j+1}^n - u_{j-1}^n)}{2\Delta x} = 0, \qquad$ for $j = 1, \ldots, N-1$

$\rightarrow$ **Upwind scheme**
$a^+ = \max(a,0), \ a^- = \min(a,0)$
$|a| = a^+ - a^-, a = a^+ + a^-$

$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a^+\frac{u_j^n - u_{j-1}^n}{\Delta x} + a^-\frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$
$\Leftrightarrow$
$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{a(u_{j+1}^n - u_{j-1}^n)}{2\Delta x} = \frac{|a|}{2\Delta x}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$
Lem. 2.3: If $|a|\frac{\Delta t}{\Delta x} \le 1$ then $E^{n+1} \le E^n$.
Example exam: We consider the equation
$\begin{cases} u_t + u_x = cu, \quad c > 0 \\ u(x,0) = u_0(x) \end{cases}$
Modify the upwind scheme to include the RHS cu:
$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) + c\Delta t u_j^n$

Weak solutions

$u \in \mathbf{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak solution of (**??**) if
$\int_{\mathbb{R}_+}\int_{\mathbb{R}}(u\varphi_t + f(u)\varphi_x)\,\mathrm{d}x\,\mathrm{d}t +$
$\int_{\mathbb{R}} u_0(x)\varphi(x,0)\,\mathrm{d}x = 0 \quad \forall\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_+)$
Every classical (strong) solution is a weak solution.
If a function is a weak solution and smooth (C^1), it is a classical solution.

Burgers' equation

Scalar conservation law with $f(u) = \frac{u^2}{2}$
$u_t + (\frac{u^2}{2})_x = 0$ (conservative form)
$u_t + uu_x = 0$ (non-conservative form)

Riemann problem

$\begin{cases} u_t + f(u)_x = 0 \\ u_0 = \begin{cases} u^L & \text{if } x < 0 \\ u^R & \text{if } x > 0 \end{cases} \end{cases}$
Characteristics:
$\begin{cases} \dot{x}(t) = u(x(t),0) \\ x(0) = x_0 \end{cases}$
$\Rightarrow \begin{cases} \text{if } x_0 < 0: & x(t) = x_0 + u^L t \\ \text{if } x_0 > 0: & x(t) = x_0 + u^R t \end{cases}$
$u^L > u^R \rightarrow$ characteristics intersect $\rightarrow$ **shock**
$u^L < u^R \rightarrow$ domains without characteristics $\rightarrow$ **rarefaction**

Shockwave: $u^L > u^R$

Rankine-Huginot condition
If u is a weak solution and u is smooth except for a jump discontinuity $\Sigma = \{x : x = \sigma(t)\}$, then away from Σ, u is a classical solution.
At Σ,
$f(u^R(t)) - f(u^L(t)) = \dot{\sigma}(u^R(t) - u^L(t))$
$\Leftrightarrow \dot{\sigma} = \frac{f(u^R(t)) - f(u^L(t))}{u^R(t) - u^L(t)}$,
with u^L, u^R being the left, right limits at the jump.

$\rightarrow$ Use to find expression for $\sigma(t)$.

$u(x,t) = \begin{cases} u^L & \text{if } x < \dot{\sigma}t \\ u^R & \text{if } x > \dot{\sigma}t \end{cases}$

Rarefaction wave: $u^L < u^R$

Solutions obtained by RH don't satisfy entropy conditions in this case.
Self-similar solution
If $f(u)$ is convex, $u(x,t) = f'^{-1}(\frac{x}{t})$ in points where there are no characteristics.
$u(x,t) = \begin{cases} u^L & \text{if } x \le f'(u^L)t \\ f'^{-1}(\frac{x}{t}) & \text{if } f'(u^L)t < x < f'(u^R)t \\ u^R & \text{if } x \ge f'(u^R)t \end{cases}$
(Just linearly connect the two constant regimes)

Combined shock/rarefaction

In the case of multiple discontinuities, a rarefaction wave can catch up to a shock (or vice versa).
Treat that point in time as IC for a new Riemann problem.
Take care when using RH: Insert analytical expression for u^R/u^L, set $x = \sigma(t)$ and solve ODE.
S5A1a):
$t = \frac{1}{2}t + 1 \Leftrightarrow t = 2$
$\sigma'(t) = \frac{f(u_r(\sigma(t),t)) - f(u_\ell(\sigma(t),t))}{u_r(\sigma(t),t) - u_\ell(\sigma(t),t)} = \frac{1}{2}\frac{\sigma(t)}{t}$
$\Rightarrow \begin{cases} \sigma'(t) = \frac{1}{2}\frac{\sigma(t)}{t} \\ \sigma(2) = 2 \end{cases} \quad \text{(solve)} \Rightarrow \quad \sigma = \sqrt{2t}$
$u(x,t) = \begin{cases} 0, & x \le f'(u_\ell)t = 0 \\ \frac{x}{t}, & f'(u_\ell)t = 0 < x \le \sigma = \sqrt{st} \\ 0, & x > \sigma = \sqrt{2t};; \quad \text{for } \mathbf{t > 2}. \end{cases}$

Entropy condition

For some IC ($u^L > u^R \rightarrow$ rarefaction), there is a region without characteristics. In this case, we can find infinitely many weak solutions for the same IC. There, the entropy condition imposes a unique solution.

'Information cannot be created, only lost' $\rightarrow$ characteristics can only flow into a shock $\leftrightarrow$ characteristics can't intersect backwards for $t > 0$

Lax entropy condition

If $f(u)$ is strictly convex, the entropy condition states
$u^R(t) \le u^L(t) \Leftrightarrow f'(u^R) \le f'(u^L)$

General entropy condition

By considering a viscous problem
$\begin{cases} u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon; \qquad \varepsilon u_{xx}^\varepsilon = \text{viscous term} \\ u^\varepsilon(x,0) = u_0(x) \\ 0 < \varepsilon \ll 1 \end{cases}$
and functions
Entropy func $S : \mathbb{R} \mapsto \mathbb{R}$,
any strictly convex function
Entropy flux $\ Q : Q(u) = \int_0^u f'(\tau)S'(\tau)\,\mathrm{d}\tau$
we find $S(u^\varepsilon)_t + Q(u^\varepsilon)_x \le \varepsilon S(u^\varepsilon)_{xx}$.
If we let $\varepsilon \rightarrow 0$, we get the entropy condition for general initial data for scalar conservation laws:
$S(u)_t + Q(u)_x \le 0$
this is calles entropy inequality, where S is entropy fct and Q is entropy flux.
Def. 3.7: An entropy solution $u(x,t)$ for a general conservation law (101)

- $u \in \mathbf{L}^\infty(\mathbb{R} \times \mathbb{R}_+)$
- u is a weak solution
- u satisfies $\int_{\mathbb{R}}\int_{\mathbb{R}_+}\big(S(u)\varphi_t + Q(u)\varphi_x\big)\,\mathrm{d}x\,\mathrm{d}t \ge 0$ $\forall\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_+), \varphi \ge 0$.

If $f(u)$ is strictly convex, entropy solutions satisfy the Lax entropy condition.
Entropy solutions exist and are unique.

Bounds on entropy solutions
Integrate over $\mathbb{R}$:
$\rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} S(u)\,\mathrm{d}x \le 0$
$\rightarrow$ Choose $S = \frac{|u|^p}{p}$
$\Rightarrow \|u(\cdot,t)\|_{\mathbf{L}^p} \le \|u_0\|_{\mathbf{L}^p}$
$\rightarrow$ Maximum principle for $p \rightarrow \infty$

Oscillations of entropy solutions
Multiply both sides of the viscous problem with $(-\text{sign}(u_x)_x)$ and integrate over space $\rightarrow$
$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|u_x|\,\mathrm{d}x \le 0 \Rightarrow \int_{\mathbb{R}}|u_x(x,t)|\,\mathrm{d}x \le \int_{\mathbb{R}}|u_0'(x)|\,\mathrm{d}x$

Fourier law: $F(u) = -k\boldsymbol{\nabla}u$.

Rem.: How to show that a result is a weak solution?
Don't have to look at the integral formulation.

- check smoothness
- check RH at shock.

Finite volume methods

Dealing with discontinuous functions → work with averages instead of point values.
Discretize domain into cells

Points	$\Delta x = \frac{x_R - x_L}{N+1}$; $t^n = n\Delta t$
(N+2)	$x_j = x_L + (j + 1/2)\Delta x$
	$x_{j-1/2} = x_j - \Delta x/2 = x_L + j\Delta x$
Cell	$\mathcal{C}_j = [x_{j-1/2}, x_{j+1/2})$
Cell avgs	$U_j^n \approx \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^n)\,\mathrm{d}x$
Fluxes	$\tilde{F}^n_{x_{j+1/2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u^L(x_{j+1/2},t))\,\mathrm{d}t$
	$\tilde{F}^n_{x_{j-1/2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u^R(x_{j-1/2},t))\,\mathrm{d}t$

The integral form of the conservation law is then

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\tilde{F}^n_{x_{j+1/2}} - \tilde{F}^n_{x_{j-1/2}}\right) \tag{18}$$

<u>*Information flow*</u>:
Information propagates from left to right:
$\partial_x u = \frac{u_j^k - u_{j-1}^k}{\Delta x}$

Godunov method

There is a Riemann problem at each cell interface:

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,t^n) = \begin{cases} U_j^n & \text{if } x < x_{j+1/2} \\ U_{j+1}^n & \text{if } x > x_{j+1/2} \end{cases} \end{cases} \tag{19}$$

Solutions to Riemann problems are self-similar, so the solution can be written as:
$\bar{u}_j(x,t) = \bar{u}_j\left(\frac{x - x_{j+1/2}}{t - t^n}\right)$
At the cell interface, $\frac{x - x_{j+1/2}}{t - t^n} \equiv 0$, so $\bar{u}_j(x_{j+1/2}, t)$ is constant and so is the flux $f(u(x_{j+1/2},t)) = f(\bar{u}_j(0))$.
$f(\bar{u}_j^R(0)) = f(\bar{u}_j^L(0))$ holds in the case of rarefaction ($\bar{u}_j$ is continuous) and shock (RH with stationary shock), so the Riemann flux $F^n_{j+1/2}$ is well defined, $F^n_{j+1/2} = \tilde{F}^n_{j+1/2}$ and (23) is the standard form of a finite volume scheme for conservation laws.

Maximum wave speed gives CFL-condition: $\max_j \left|f'(U_j^n)\right| \frac{\Delta t}{\Delta x} \le \frac{1}{2}$

Godunov flux

Explicitly calculate the fluxes
$F^n_{j+1/2} = F(U_j^n, U_{j+1}^n) =$

$$= \begin{cases} \min\limits_{\Theta \in [U_j^n, U_{j+1}^n]} f(\Theta) & \text{if } U_j^n \le U_{j+1}^n \\ \max\limits_{\Theta \in [U_{j+1}^n, U_j^n]} f(\Theta) & \text{if } U_j^n \ge U_{j+}^n \end{cases}$$

If $f(u)$ has no $\substack{\text{maxima}\\\text{minima}}$ exactly one $\substack{\text{minimum}\\\text{maximum}}$ at ω in an interval (a,b) "⇔" $f(u)$ is $\substack{\text{convex}\\\text{concave}}$:
(ω is a minimum/maximum)

$$F^n_{j+1/2} = \substack{\max\\\min}\left\{ f\left(\substack{\max\\\min}\{U_j^n, \omega\}\right), f\left(\substack{\min\\\max}\{U_{j+1}^n, \omega\}\right)\right\}$$

The Godunov flux gives good results, but it is expensive to compute for general fluxes and relies on an explicit formula for solutions of the Riemann problem (not available for more complex problems than scalar conservation laws).
Ex. The flux in the case of the Burgers has a unique minimum at $u = 0$ (convex).

Roe's scheme

Linearize the Riemann problem
Replace $f(u)_x$ by $\hat{A}_{j+1/2}U_x$ in in (24).
For example
$\hat{A}_{j+1/2} = f'\left(\frac{U^n_{j+1/2} - U^n_{j-1/2}}{2}\right)$
or Roe's choice:

$$\hat{A}_{j+1/2} = \begin{cases} \frac{f(U^n_{j+1}) - f(U^n_j)}{U^n_{j+1} - U^n_j} & \text{if } U^n_{j+1} \ne U^n_j \\ f'(U^n_j) & \text{if } U^n_{j+1} = U^n_j \end{cases}$$

The explicit solution of the linearized Riemann problem yields:

$$F^{n,\mathrm{Roe}}_{j+1/2} = \begin{cases} f(U^n_j) & \text{if } \hat{A}_{j+1/2} \ge 0 \\ f(U^n_{j+1}) & \text{if } \hat{A}_{j+1/2} < 0 \end{cases}$$

Roe's scheme is fast and yields correct results for shocks, but gives solutions for rarefaction which don't satisfy the entropy condition. This is because the linearized Riemann problem is solved by just one wave (correct for shock, should be two for rarefaction)
Rem.: • Roe computes weak sol's not *the* entropy sol.
• Since the Roe flux approximates the local Riemann problems using only one wave, it should *not* resolve the rarefaction wave correctly and hence both the L^1 and L^∞ errors should *not* converge.

Central schemes

The solution of the Riemann problem is approximated by a left- (speed $s^l_{j+1/2}$) and a right-travelling ($s^r_{j+1/2}$) wave.
This gives a solution

$$u(x,t) = \begin{cases} U_j^n & \text{if } x < s^l_{j+1/2}t \\ U^*_{j+1/2} & \text{if } s^l_{j+1/2}t \le x \le s^r_{j+1/2}t \\ U^n_{j+1} & \text{if } x > s^r_{j+1/2}t \end{cases}$$

With RH, we can determine the intermediate flux $f^*_{j+1/2}$ to be

$$f^*_{j+1/2} = \begin{cases} \frac{s^r_{j+1/2}f(U^n_j) - s^l_{j+1/2}f(U^n_{j+1}) + s^l_{j+1/2}s^r_{j+1/2}(U^n_{j+1} - U^n_j)}{s^r_{j+1/2} - s^l_{j+1/2}} \\ \quad \text{if } s^r_{j+1/2} \ne -s^l_{j+1/2} \\ \frac{f(U^n_j) + f(U^n_{j+1}) - s_{j+1/2}(U^n_{j+1} - U^n_j)}{2} \\ \quad \text{if } s^r_{j+1/2} = -s^l_{j+1/2} = s_{j+1/2} \end{cases}$$

The numerical flux is then

$$F^n_{j+1/2} = \begin{cases} f(U^n_j) & \text{if } s^l_{j+1/2}, s^r_{j+1/2} \ge 0 \\ f^*_{j+1/2} & \text{if } s^l_{j+1/2} \le 0 \le s^r_{j+1/2} \\ f(U^n_{j+1}) & \text{if } s^l_{j+1/2}, s^r_{j+1/2} \le 0 \end{cases}$$

<u>Choice of wavespeeds</u>

Lax-Friedrichs	$s^r_{j+1/2} = \frac{\Delta x}{\Delta t}$, $s^l_{j+1/2} = -s^r_{j+1/2}$ Maximum speed without waves interacting
Rusanov	$s^r_{j+1/2} = \max\{\lvert f'(U^n_j)\rvert, \lvert f'(U^n_{j+1})\rvert\}$, $s^l_{j+1/2} = -s^r_{j+1/2}$ Local max. speed for both waves

Both choices lead to approximations of the entropy solution, but are diffusive (LF more so than Rusanov).

Lax-Friedrichs flux

$F^n_{j+1/2} = F^{\mathrm{LxF}}(u^n_j, u^n_{j+1})$
$= \frac{f(u^n_j) + f(u^n_{j+1})}{2} - \frac{\Delta x}{2\Delta t}(u^n_{j+1} - u^n_j)$

Rusanov flux

$F^n_{j+1/2} = F^{\mathrm{Rus}}(u^n_j, u^n_{j+1}) = \frac{f(u^n_j) + f(u^n_{j+1})}{2}$
$- \frac{\max(|f'(u^n_j)|, |f'(u^n_{j+1})|)}{2}(u^n_{j+1} - u^n_j)$

Engquist-Osher scheme

Flux: $F^n_{j+1/2} = F^{\mathrm{EO}}(u^n_j, u^n_{j+1})$
$= \frac{f(u^n_j) + f(u^n_{j+1})}{2} - \frac{1}{2}\int_{u^n_j}^{u^n_j j+1} |f'(\theta)|\,\mathrm{d}\theta$
When the flux fuct has a single minimum at a point ω and no maxima, the EO flux can be explicitly computed as
$F^{\mathrm{EO}}(u^n_j, u^n_{j+1})$
$= f(\max(u^n_j, \omega)) + f(\min(u^n_{j+1}, \omega)) - f(\omega)$
EO scheme for convex fluxes:
$F^{\mathrm{EO}}(u^n_j, u^n_{j+1}) = f^+(u^n_j) + f^-(u^n_{j+1})$, where $f^+(u) = f(\max(u,\omega))$, $f^-(u) = f(\min(u,\omega))$ positive (increasing) and negative (decreasing), i.e. EO scheme is a *flux splitting scheme*.
Rem.: Godunov & EO best in this case.

Analysis

Conservative, consistent, monotone schemes

Rewrite (23) in update form:
$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}(F(U^n_{j+1}, U^n_j) - F(U^n_j, U^n_{j-1}))$
$= H(U^n_{j+1}, U^n_j, U^n_{j-1})$

- A scheme is conservative ⇔ $\sum_j U_j^{n+1} = \sum_j U_j^n$
- A scheme is consistent ⇔ $F(a,a) = f(a) \Rightarrow H(a,a,a) = a$
- A scheme is monotone ⇔ $H(a,b,c)$ is a non-decreasing function in each of its arguments ⇔ $\frac{\partial H}{\partial a} \ge 0$, $\frac{\partial H}{\partial b} \ge 0$, $\frac{\partial H}{\partial c} \ge 0$

	Conservative	Consistent	Monotone
Godunov	✓	✓	✓
Roe	✓	✓	✗
LF	✓	✓	✓
Rusanov	✓	✓	✓
Central	✓	✓	✗

(under the CFL condition)
<u>*Thm. 4.5. (Conservative schemes)*</u> Assume that $H(0,...,0) = 0$. Then $U_j^{n+1} = H(u^n_{j-n}, ..., u^n_{j+p})$ is conservative iff there exists a fct $F^n_{j+1/2} = F(u^n_{j-p+1}, ..., u^n_{j+p})$ s.t. $U_j^{n+1} = H(u^n_{j-n}, ..., u^n_{j+p})$ can be written in the FV form $u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F^n_{j+1/2} - F^n_{j-1/2})$.
<u>*Lem. 4.8. (Monotone schemes)*</u> *A FV scheme $u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F^n_{j+1/2} - F^n_{j-1/2})$ with a two-point flux $F = F(a,b)$ is monotone* if F is a Lipschitz continuous, differentiable fct in both arguments, non-decreasing in its first argument and non-increasing in its second argument. The following holds too,
$a \mapsto F(a,b)$ is non-decreasing for fixed b,
$b \mapsto F(a,b)$ is non-increasing for fixed a,
and the following CFL-type condition holds:
$|\frac{\partial F}{\partial x_1}(b,c)| + |\frac{\partial F}{\partial x_2}(a,b)| \le \frac{\Delta x}{\Delta t}$.

Maximum principle

A monotone, consistent, conservative scheme satisfies the maximum principle $\min\{U_{j-1}^n, U_j^n, U_{j+1}^n\} \leq U_j^{n+1} \leq \max\{U_{j-1}^n, U_j^n, U_{j+1}^n\}$ in particular, we have $\min_k(u_k^0) \leq u_j^n \leq \max_k(u_k^0) \quad \forall n, j.$

Entropy inequalities & L^p bounds

Def.: The Crandall-Majda numerical entropy flux:
$Q_{j+1/2}^n = Q(u_j^n, u_{j+1}^n) = F(u_j^n \vee k, u_{j+1}^n \vee k) - F(u_j^n \wedge k, u_{j+1}^n \wedge k)$,
where $a \vee b = \max(a,b), \quad a \wedge b = \min(a,b)$.
Lem. 4.10 (Crandall-Majda) Let u_j^n be an approximate solution computed by a consistent and monotone three-point scheme $u_j^{n+1} = H(u_{j-1}^n, u_j^n, u_{j+1}^n)$. Then u satisfies the discrete entropy inequality
$|u_j^{n+1} - k| - |u_j^n - k| + \frac{\Delta t}{\Delta x}(Q_{j+1/2}^n - Q_{j-1/2}^n) \leq 0 \quad \forall n, j.$

Total variation principle

Def. (Lipschitz flux): $\frac{F(u_j^n, u_j^n) - F(u^n, u_{j+1}^n)}{u_{j+1}^n - u_j^n} \leq |\frac{\partial F}{\partial b}|$
Lem. 4.13 (TV bounds): Any consistent monotone three-point FV scheme $u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F_{j+1/2}^n - F_{j-1/2}^n)$ is TVD under the CFL condition $|\frac{\partial F}{\partial x_1}(b,c)| + |\frac{\partial F}{\partial x_2}(a,b)| \leq \frac{\Delta x}{\Delta t}$.
Total variation: $TV(u) = \int_{\mathbb{R}} |u_x| \,\mathrm{d}x \hat{=} \sum_j |u_{j+1}^n - u_j^n|$
Rewrite (23) in increment form:
$U_j^{n+1} = U_j^n + C_{j+1/2}^n(U_{j+1}^n - U_j^n) - D_{j-1/2}^n(U_j^n - U_{j-1}^n)$
$C_{j+1/2}^n = \frac{\Delta t}{\Delta x} \frac{f(U_j^n) - F(U_{j+1}^n, U_j^n)}{U_{j+1}^n - U_j^n}$
$D_{j+1/2}^n = \frac{\Delta t}{\Delta x} \frac{f(U_{j+1}^n) - F(U_{j+1}^n, U_j^n)}{U_{j+1}^n - U_j^n}$

Harten's Lemma
For a scheme in increment form, as long as
$C_{j+1/2}^n, D_{j+1/2}^n \geq 0 \quad \forall j$
$C_{j+1/2}^n + D_{j+1/2}^n \leq 1 \quad \forall j$
then the scheme is TVD, i.e.
$TV(U^{n+1}) \leq TV(U^n)$

Schemes which are monotone, conservative and consistent satisfy the conditions for Harten's lemma.

Convergence result

Approximate solutions with a pw constant function:
$U^\Delta(x,t) = u_j^n \quad$ for $(x,t) \in [x_{j-1/2}, x_{j+1/2}) \times [t^n, t^{n+1})$.
Thm. 4.14 (Lax-Wendroff) Let u_j^n be approximate solutions of $u_t + f(u)_x = 0$, computed by a conservative and consistent FV scheme $u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F_{j+1/2}^n - F_{j-1/2}^n)$ with a differentiable (or Lipschitz) numerical flux fct F, where u_j^0 is given by $u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,\mathrm{d}x$. Assume that $u_0 \in L^\infty(\mathbb{R})$ and that the approsimating fcts u^Δ

- are uniformly bounden, i.e. $\|u^\Delta\|_{L^\infty(\mathbb{R}\times\mathbb{R})} \leq C$ for some const. $C > 0$;
- converge in $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R})$ and almost ecerywhere to some fct u.

Then u is a weak solution of $u_t + f(u)_x = 0$, with initial data u_0.

Boundary conditions

Solve PDE on domain $[x_L, x_R]$ with mesh $x_{1/2} = x_L$ and $x_{N+1/2} = x_R$ ($N+1$ points). Introduce ghost cells:
$\mathcal{C}_0 = [x_L - \Delta x, x_L),\ \mathcal{C}_{N+1} = [x_R, x_R + \Delta x)$
Dirichlet BC
$U_0^n = g(x_L, t^n),\ U_{N+1}^n = g(x_R, t^n)$
Periodic BC:
$U_0^n = U_N^n,\ U_{N+1}^n = U_1^n$
Non-reflecting Neumann BC: (Artificial BC)
$U_0^n = U_1^n,\ U_{N+1}^n = U_N^n$

Miscellaneous

Finite differences

Forward diff.: $u'(t^n) \approx [\mathbf{D}_t^+ u]^n = \frac{u^{n+1} - u^n}{\Delta t}$
$\rightarrow \mathbf{R}^n = \frac{\Delta t}{2} u''(t_n) + \mathcal{O}(\Delta t^2)$
Backward diff.: $u'(t^n) \approx [\mathbf{D}_t^- u]^n = \frac{u^n - u^{n-1}}{\Delta t}$
$\rightarrow \mathbf{R}^n = -\frac{\Delta t}{2} u''(t_n) + \mathcal{O}(\Delta t^2)$
Central diff.: $u'(t^n) \approx [\mathbf{D}_t u]^n = \frac{u^{n+\frac{1}{2}} - u^{n-\frac{1}{2}}}{\Delta t}$
$\rightarrow \mathbf{R}^n = \frac{\Delta t^2}{24} u'''(t_n) + \mathcal{O}(\Delta t^4)$
Central diff. for 2nd derivative:
$u''(t^n) \approx [\mathbf{D}_t^- \mathbf{D}_t^+ u]^n = [\mathbf{D}_t^+ \mathbf{D}_t^- u]^n = [\mathbf{D}_t \mathbf{D}_t u]^n = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}$
$\rightarrow \mathbf{R}^n = \frac{\Delta t^2}{12} u^{(4)}(t_n) + \mathcal{O}(\Delta t^4)$

$\mathbf{R}^n = [\mathbf{D}_t u]^n - u'(t_n)$,
insert Taylor expansion of $[\mathbf{D}_t u]^n$
Crank-Nicolson (second-order convergence in time!) if given a PDE:
$\frac{\partial u}{\partial t} = \mathbf{F}\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)$
and the space discretisation (e.g. upwind):
$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mathbf{F}_i^n\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)$ (forward Euler)
$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mathbf{F}_i^n\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)$ (backward Euler)
Then we the following holds:
$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left[\mathbf{F}_i^{n+1}\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) + \mathbf{F}_i^n\left(u, x, t, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)\right]$
$Err^{(k)}(\Delta x, \Delta t) = C_1 \Delta t^2 + C_2 \Delta x$
Example with linear transport equation (upwind for space):
$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left(-a(x_i)\frac{u_i^n - u_{i-1}^n}{\Delta x} + -a(x_i)\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x}\right)$
Trick:
$\frac{\partial \tilde{u}}{\partial r}(r_{i+1/2}, t^n) \approx \frac{\tilde{D}_x^+ u_{i+1/2}^n + \tilde{D}_x^- u_{i+1/2}^n}{2}$
$= \frac{1}{2}\left(\frac{u_{i+1}^n - u_{i+1/2}^n}{\frac{\Delta r}{2}} + \frac{u_{i+1/2}^n - u_i^n}{\frac{\Delta r}{2}}\right) = \frac{u_{i+1}^n - u_i^n}{\Delta r}$

Hilbert, Sobolev and L-Spaces

$\mathbf{C}_0^k((0,1)) = \{g \mid g \in \mathbf{C}^k((0,1) \cap \mathbf{C}([0,1])$ s.t. $g(0) = g(1) = 0\}$
Hilbert:
A Hilbert space is a complete (every converging sequence of elements of $\mathbf{V}$ converges to an element of $\mathbf{V}$) vector space $\mathbf{V}$ with properties:

- Addition is commutative, associative and has identity and inverse elements
- Scalar multiplication is (associative), distributive and has an identity element

with an inner product:
$(\cdot,\cdot) : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$
$v, w \mapsto (v, w)$
which has the properties:

- Symmetry: $(v,w) = (w,v)$
- Bilinearity: $(\alpha v + \beta u, w) = \alpha(v,w) + \beta(u,w)$
- $(v,w) = 0 \Leftrightarrow v = 0 \vee w = 0$ and $(v,w) \geq 0 \forall v, w \in \mathbf{V}$
- Induces a norm: $\|v\|_{\mathbf{V}} = \sqrt{(v,v)}$

Lebesgue: (Notion: $L^2(\Omega) \rightarrow \mathcal{C}_{pw}^0(\Omega)$)
$\mathbf{L}^1(\Omega)$-norm: $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$
$\mathbf{L}^2(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : \int_\Omega |u(x)|^2 \,\mathrm{d}x < \infty\}$
$(u,v)_{L^2} = \int_\Omega u(x)v(x)\,\mathrm{d}x$
$(\|v\|_0 :=)\|v\|_{L^2(\Omega)} := \left(\int_\Omega |v(x)|^2 \,\mathrm{d}x\right)^{1/2}$
$\mathbf{L}^\infty(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : \sup_{x\in\Omega} |u(x)| < \infty\}$
$\|u\|_{\mathbf{L}^\infty} = \sup_{x\in\Omega} |u(x)|$
$\|u\|_\infty = \max_{x\in\Omega} |u(x)|$
$\mathbf{L}^p(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : \int_\Omega |u(x)|^p \,\mathrm{d}x < \infty\}$
$\|u\|_{\mathbf{L}^p} = \left(\int_\Omega u^p \,\mathrm{d}x\right)^{\frac{1}{p}}$ (no inner product for $p \neq 2$)
Sobolev: (Notion: $H_0^1(\Omega) \rightarrow \mathcal{C}_{pw}^1(\Omega)$)
$\mathbf{H}^1((0,1)) = \{u : \Omega \rightarrow \mathbb{R} : u' \in \mathbf{L}^2\}$
$(u,v)_{\mathbf{H}^1} = \int_\Omega uv + u'v' \,\mathrm{d}x$
$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|u\|_{H_0^1(\Omega)}^2$
$\mathbf{H}_0^1(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : u' \in \mathbf{L}^2, u(0) = u(1) = 0\}$
$(u,v)_{\mathbf{H}_0^1} = \int_\Omega u'(x)v'(x)\,\mathrm{d}x$
$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |u'|^2 \,\mathrm{d}x\right)^{1/2}$
$\mathbf{H}^k(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : u', u'', ..., u^{(k)} \in \mathbf{L}^2\}$
$(u,v)_{\mathbf{H}^k} = \sum_{i=0}^k \left(u^{(i)}, v^{(i)}\right)_{\mathbf{L}^2}$
Other norms:
$\|f\|_{\Delta x,\infty} := \sup_{1\leq j\leq N} |f_j| = \sup_{1\leq j\leq N} |f(x_j)|$
$\|a\|_{C^1} := \sup_{y\in\mathbb{R}} |a(y,t)| + \sup_{y\in\mathbb{R}} |a_x(y,t)|$

$\mathbf{H}^0 = \mathbf{L}^2$
$\mathbf{L}^2$, $\mathbf{H}^k$ are Hilbert spaces
$u \in \mathbf{H}_0^1 \Rightarrow u \in \mathbf{L}^2$ (Poincaré)

Forms

Linear:
$\boldsymbol{l} : \mathbf{V} \rightarrow \mathbb{R}$
$v \mapsto \boldsymbol{l}(v)$
$\boldsymbol{l}(\alpha v + \beta w) = \alpha \boldsymbol{l}(v) + \beta \boldsymbol{l}(w)$

Bilinear:
$\boldsymbol{a} : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$
$v, w \mapsto \boldsymbol{a}(v,w)$
$\boldsymbol{a}(\alpha v + \beta u, w) = \alpha \boldsymbol{a}(v,w) + \beta \boldsymbol{a}(u,w)$
$\boldsymbol{a}(v, \alpha u + \beta w) = \alpha \boldsymbol{a}(v,w) + \beta \boldsymbol{a}(v,w)$
Need not be symmetric!
$\boldsymbol{a}(\cdot,\cdot)$ symmetric $\Leftrightarrow \boldsymbol{a}(v,w) = \boldsymbol{a}(w,v) \forall v, w \in \mathbf{V}$

Compact support / Smoothness

The *support* is:
$\mathrm{supp}(u) = \{x \in \Omega : u(x) \neq 0\}$
A set is compact if it is both closed($[a,b]$, not (a,b)) and bounded.
φ is continuously differentiable with compact support $\equiv \varphi \in \mathcal{C}_c^1$
Smoothness: $f \in \mathcal{C}^\infty$, then f is smooth.

FEM basis functions

On the reference triangle k $\{a_1, a_2, a_3\} = \{(0,0), (1,0), (0,1)\}$, the shape functions are:
$M = \{x_0 = a, x_1, ..., x_N, x_{N+1} = b\}$.

$$1\mathrm{D}\ (p=1, S_1^0(M)): \begin{cases} \hat{\lambda}_0(\mathbf{x}) = 1 - x - y \\ \hat{\lambda}_1(\mathbf{x}) = x \\ \hat{\lambda}_2(\mathbf{x}) = y \end{cases}$$

$$\text{Gradients: } \begin{cases} \boldsymbol{\nabla}\hat{\lambda}_0(\mathbf{x}) = (-1,-1)^T \\ \boldsymbol{\nabla}\hat{\lambda}_1(\mathbf{x}) = (1,0)^T \\ \boldsymbol{\nabla}\hat{\lambda}_2(\mathbf{x}) = (0,1)^T \end{cases}$$

$$2\mathrm{D}\ (p=2, S_2^0(M)): \begin{cases} \hat{\varphi}_0(\mathbf{x}) = (2\hat{\lambda}_0(\mathbf{x}) - 1)\hat{\lambda}_0(\mathbf{x}) \\ \hat{\varphi}_1(\mathbf{x}) = (2\hat{\lambda}_1(\mathbf{x}) - 1)\hat{\lambda}_1(\mathbf{x}) \\ \hat{\varphi}_2(\mathbf{x}) = (2\hat{\lambda}_2(\mathbf{x}) - 1)\hat{\lambda}_2(\mathbf{x}) \\ \hat{\varphi}_3(\mathbf{x}) = 4\hat{\lambda}_0(\mathbf{x})\hat{\lambda}_1(\mathbf{x}) \\ \hat{\varphi}_4(\mathbf{x}) = 4\hat{\lambda}_1(\mathbf{x})\hat{\lambda}_2(\mathbf{x}) \\ \hat{\varphi}_5(\mathbf{x}) = 4\hat{\lambda}_0(\mathbf{x})\hat{\lambda}_2(\mathbf{x}) \end{cases}$$

$$\begin{cases} \boldsymbol{\nabla}\hat{\varphi}_0(\mathbf{x}) = (4\hat{\lambda}_0(\mathbf{x}) - 1)\boldsymbol{\nabla}\hat{\lambda}_0(\mathbf{x}) \\ \boldsymbol{\nabla}\hat{\varphi}_1(\mathbf{x}) = (4\hat{\lambda}_1(\mathbf{x}) - 1)\boldsymbol{\nabla}\hat{\lambda}_1(\mathbf{x}) \\ \boldsymbol{\nabla}\hat{\varphi}_2(\mathbf{x}) = (4\hat{\lambda}_2(\mathbf{x}) - 1)\boldsymbol{\nabla}\hat{\lambda}_2(\mathbf{x}) \\ \boldsymbol{\nabla}\hat{\varphi}_3(\mathbf{x}) = 4(\boldsymbol{\nabla}\hat{\lambda}_0(\mathbf{x})\hat{\lambda}_1(\mathbf{x}) + \hat{\lambda}_0(\mathbf{x})\boldsymbol{\nabla}\hat{\lambda}_1(\mathbf{x})) \\ \boldsymbol{\nabla}\hat{\varphi}_4(\mathbf{x}) = 4(\boldsymbol{\nabla}\hat{\lambda}_1(\mathbf{x})\hat{\lambda}_2(\mathbf{x}) + \hat{\lambda}_1(\mathbf{x})\boldsymbol{\nabla}\hat{\lambda}_2(\mathbf{x})) \\ \boldsymbol{\nabla}\hat{\varphi}_5(\mathbf{x}) = 4(\boldsymbol{\nabla}\hat{\lambda}_0(\mathbf{x})\hat{\lambda}_2(\mathbf{x}) + \hat{\lambda}_0(\mathbf{x})\boldsymbol{\nabla}\hat{\lambda}_2(\mathbf{x})) \end{cases}$$

where we have the points: $p_1 = a_1, p_2 = a_2, p_3 = a_3, p_4 = \frac{1}{2}(a_1+a_2), p_5 = \frac{1}{2}(a_2+a_3), p_5 = \frac{1}{2}(a_1+a_3)$.

Lem. Dimension of spaces of polynomials

$dim(\mathcal{P}_p(\mathbb{R}^d)) = \binom{d+p}{p}, \quad \forall p \in \mathbb{N}_0, d \in \mathbb{N}, \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Integration by parts

$\int_b^a fg'\,\mathrm{d}x = [fg]_b^a - \int_b^a f'g\,\mathrm{d}x$

General product rule:

$div(\vec{f}g) = \vec{f}\cdot grad(g) + g\,div(\vec{f})$

Divergence: $div(\mathbf{F}) = \boldsymbol{\nabla}\cdot\mathbf{F}$

Summation by parts

$\sum_{j=1}^{N} v_j \mathbf{D}^+ w_j = v_{N+1}w_{N+1} - v_0 w_0 - \sum_{j=0}^{N} w_j \mathbf{D}^- v_j$

Identity:

$D_x^- D_x^+ u_j^n = \frac{1}{\Delta x}(D_x^+ u_j^n - D_x^- u_j^n)$

Symmetric case:

$\sum_{j=1}^N w_j^n D_x^- D_x^+ w_j^{n+1}$

$= -\sum_{j=0}^N (D_x^+ w_j^n)(D_x^+ w_j^{n+1})$

Discrete Chain rule

$w_j^n \mathbf{D}_t^+ w_j^n = \frac{1}{2}\mathbf{D}_t^+\left((w_j^n)^2\right) - \frac{\Delta t}{2}\left(\mathbf{D}_t^+ w_t^n\right)^2$

$w_j^n \mathbf{D}_t^- w_j^n = \frac{1}{2}\mathbf{D}_t^-\left((w_j^n)^2\right) - \frac{\Delta t}{2}\left(\mathbf{D}_t^- w_t^n\right)^2$

Taylor

$f(x\pm h) = \sum_{j=0}^{\infty}\frac{(\pm h)^j}{j!}\frac{\mathrm{d}^j f}{\mathrm{d}x^j}$

$Tf(x;a) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$

Cauchy-Schwarz inequality

$|\langle u,w\rangle|^2 \le \langle u,u\rangle\cdot\langle w,w\rangle$

$\sum_{j=1}^N u_j w_j \le \left(\sum_{j=1}^N u_j^2\right)^{1/2}\left(\sum_{j=1}^N w_j^2\right)^{1/2}$

$\int_a^b u(\tau)w(\tau)\,\mathrm{d}\tau \le \left(\int_a^b u^2(\tau)\,\mathrm{d}\tau\right)^{1/2}\left(\int_a^b w^2(\tau)\,\mathrm{d}\tau\right)^{1/2}$

$|a(u,v)| \le (a(u,v))^{1/2}(a(u,v))^{1/2}$

Poincaré inequality

$|u\|_{L^2} \le \|u\|_{H_0^1} \Leftrightarrow$

$\left(\int|u(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \le \left(\int|u'(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$

Gronwall's inequality

Let $\beta(t)$ be continuous and $u(t)$ differentiable on $[a,b]$ and assume

$u'(t) \le \beta(t)u(t) \quad \forall t\in[a,b]$.

Then:

$u(t) \le u(a)\exp\left(\int_a^t \beta(\tau)\,\mathrm{d}\tau\right) \quad \forall t\in[a,b]$

Gauss/Green formula

For $\Omega\in\mathbb{R}^d$ and $\mathbf{j},\mathbf{F}\in\mathcal{C}^1(\Omega)^d, v\in\mathcal{C}^1(\Omega)$:

Gauss: $\int_\Omega \nabla\cdot\mathbf{F}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$

Green: $-\int_\Omega \nabla\cdot\mathbf{j}v\,\mathrm{d}\mathbf{x} = -\int_{\partial\Omega}\mathbf{j}\cdot\mathbf{n}v\,\mathrm{d}S + \int_\Omega \mathbf{j}\cdot\nabla v\,\mathrm{d}\mathbf{x}$

$\int_\Omega \mathbf{j}\cdot grad(v)\,\mathrm{d}\mathbf{x} = -\int_\Omega div(\mathbf{j})v\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\mathbf{j}\cdot\mathbf{n}v\,\mathrm{d}S$

$\rightsquigarrow \quad \int_\Omega \underbrace{div(u)}_{''v''}\,div(\underbrace{v}_{''j''})\,\mathrm{d}\mathbf{x} \to$ Green

Laplace operator

cartesian: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

cylindrical: $\Delta = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial}{\partial z}$

Quadrature

Composite trapezoidal quadrature

$\int_a^b f(x)\,\mathrm{d}x \approx f(a)\frac{h}{2} + h\sum_{i=1}^N f(x_i) + f(b)\frac{h}{2}$

Simple trapezoidal rule:

$\int_a^b f(x)\,\mathrm{d}x \approx (b-a)\left[\frac{f(b)+f(a)}{2}\right]$

Midpoint rule:

$\int_a^b f(x)dx \approx (b-a)f(\frac{a+b}{2})$

Fundamental theorem of integral calculus

$u(x) = C + \int\limits_0^x u'(\tau)\,\mathrm{d}\tau$

$u'(y) = C + \int\limits_0^y u''(z)\,\mathrm{d}z$

Fundamental lemma of variational calculus

1D:

Let $f\in\mathcal{C}^0_{pw}\left([a,b]\right), -\infty < a < b < \infty$ satisfies:

$$\int_a^b f(\xi)v(\xi)\,\mathrm{d}\xi = 0$$

$$\forall v\in\mathcal{C}^k\left([a,b]\right), v(a) = v(b) = 0$$

for some $k\in\mathbb{N}_0$.

Then: $f\equiv 0$.

In higher dimensions:

If $f\in L^2(\Omega)$ satisfies:

$$\int_\Omega f(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0, \quad \forall v\in\mathcal{C}_0^\infty(\Omega)$$

Then: $f\equiv 0$.

Convex functions / Lipschitz

Convex functions: Let $f\in\mathcal{C}^2(\Omega)$.

Then f is convex $\Leftrightarrow f''(x)\ge 0 \quad \forall x\in\Omega$.

Def. Lipschitz: Eine Funktion $f:\Omega\subset\mathbb{R}^d\to\mathbb{R}^n$ heisst *Lipschitz stetig* mit *Lipschitzkonstante* L, falls gilt

$\|f(x)-f(y)\| \le L\|x-y\|, \quad \forall x,y\in\Omega.$

Satz Lipschitz stetig:

Eine differenzierbare Funktion $f:\Omega\to\mathbb{R}$ ist genau dann Lipschitz-stetig, wenn ihre erste Ableitung auf Ω beschränkt ist.

Identities

- Elementary inequality: $ab \le \frac{a^2}{2} + \frac{b^2}{2}$
- Simple algebraic relation: $(a-b)^2 \le 2(a^2+b^2)$
- $b(a-b) = \frac{a^2}{2} - \frac{b^2}{2} - \frac{1}{2}(a-b)^2$

Miscellaneous

Math Thm/Lem/Cor

$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d\end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$

Multiplicative trace inequality:

$\exists C = C(\Omega) > 0:$

$\|u\|^2_{L^2(\partial\Omega)} \le C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}$

$\forall u\in H^1(\Omega)$

Rem.:

$\|u\|^2_{L^2(\partial\Omega)} \le C\|u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}$

$\overset{\text{poincare}}{\le} C\|u\|_{H_0^1(\Omega)}\|u\|_{H^1(\Omega)}$

$\le C\|u\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)}$

$\Leftrightarrow \|\mathbf{u}\|_{\mathbf{L^2(\partial\Omega)}} \le \mathbf{C}\|\mathbf{u}\|_{\mathbf{H^1(\Omega)}}$.

Example from lecture:

$-\mu\Delta u + (\beta\cdot\nabla)u = f$

$\Rightarrow -\int_\Omega \mu v\Delta u\,\mathrm{d}x + \beta_1\int_\Omega vu_x\,\mathrm{d}x + \beta_2\int_\Omega vu_y\,\mathrm{d}x = \int_\Omega vf\,\mathrm{d}x$

$\bullet\beta_1\int_\Omega(\frac{u^2}{2})_x\,\mathrm{d}x + \beta_2\int_\Omega(\frac{u^2}{2})_y\,\mathrm{d}x$

$= \frac{\beta_1}{2}\int_\Omega u_x^2\,\mathrm{d}x + \frac{\beta_2}{2}\int_\Omega u_y^2 dx$

$= \frac{1}{2}\int_\Omega \beta\cdot\nabla v dx$

$\overset{\text{i.b.p.}}{=} \frac{1}{2}[-\int_\Omega div(\beta)u\,\mathrm{d}x + \int_\Omega \beta\cdot\hat{n}v\,\mathrm{d}S]$